General Relativity

Re-take Exam
28/11/2013
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Problem #1: Anti-de Sitter space

Consider the 2-dimensional spacetime

$$ds^2 = r^2 dt^2 - \frac{dr^2}{r^2}$$

defined for $r > 0$.

1. Write the components of the metric and compute the Christoffel symbols.
2. Consider the Riemann tensor with all components lower. How many independent components does it have in 2 dimensions? Compute *the independent components* of the Riemann tensor (with lower indices) for this metric.
3. Derive the equations of motion for massive particles moving in this spacetime.
4. Derive an effective potential for the motion of massive particles in this geometry. Using this potential show that particles feel a force towards smaller value of r and also that massive particles moving along geodesics can never reach the region $r \to \infty$.
5. Consider a massive particle of mass m staying at constant value of $r = r_0$. Compute the relativistic acceleration that the particle undergoes and the force needed to keep it in this orbit.

Problem #2: FRW evolution and equation of state

Consider a homogeneous isotropic Universe describe by the Robertson-Walker metric with $k = 0$ i.e.

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

Assume that the Universe is filled with a fluid described by the equation of state

$$P = w\rho$$

Here P is the pressure, ρ the density and w some constant which characterizes the type of fluid.

1. Write the FRW equations for this Universe.
2. By following steps similar to those for the matter/radiation dominated Universe, show that the density is related to the scale factor as
$$\rho = C\, a^{-3(1+w)}$$
where C is some constant.
3. Solve for the scale factor $a(t)$ and show that for late times the scale factor depends on time as
$$a(t) \approx D\, t^{\frac{2}{3(1+w)}}$$
where D is another constant.

Problem #3: Geometry of a charged black hole

Consider the metric

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with

$$f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$$

which describes the spacetime around a black hole of mass M and electric charge Q.

1. Find the conditions for the parameters M, Q in order for the equation $f(r) = 0$ to have two real solutions. We call these solutions r_+, r_- where we take our conventions such that $r_+ > r_-$. Naively the metric appears to be singular at $r = r_+$ and $r = r_-$, but we will argue that these are coordinate singularities and play the role of horizons.

2. Argue that the function f can be written as

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2}$$

3. Consider radially infalling light rays. Show that they can be describe by the equation

$$\frac{r^2}{(r - r_+)(r - r_-)} dr = -dt$$

4. In order to simplify the integration of this equation verify that we can write

$$\frac{r^2}{(r - r_+)(r - r_-)} = 1 + \left(\frac{r_+^2}{r_+ - r_-}\right)\frac{1}{r - r_+} - \left(\frac{r_-^2}{r_+ - r_-}\right)\frac{1}{r - r_-}$$

5. Using the simplification of point 4. solve the radial null equation in point 3. Define the Eddington Finkelstein time coordinate $\tilde{t}$ in which the equations of motion of infalling light rays is

$$\tilde{t} + r = \text{constant}$$

   The answer should be

$$\tilde{t} = t + \left(\frac{r_+^2}{r_+ - r_-}\right)\log(r - r_+) - \left(\frac{r_-^2}{r_+ - r_-}\right)\log(r - r_-)$$

6. Check that $d\tilde{t} = dt + (f^{-1} - 1)dr$. Using this, eliminate the variable t and write the metric in terms of the variables $\tilde{t}, r, \theta, \phi$. Show that it takes the form

$$ds^2 = f d\tilde{t}^2 - 2(1 - f)dr d\tilde{t} - (2 - f)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

   Which is regular for all $r > 0$.

7. In the $\tilde{t}, r$ coordinates, find the orientation of the lightcone as a function of r. Draw a lightcone diagram for this metric in the $\tilde{t}, r$ coordinates and explain physically what happens at the horizons $r = r_+$ and $r = r_-$.